LESSON 8: MODELING PHYSICAL SYSTEMS WITH LINEAR DIFFERENTIAL EQUATIONS

ET 438a Automatic Control Systems Technology

Learning Objectives

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After this presentation you will be able to:

- Explain what a differential equation is and how it can represent dynamics in physical systems.
- Identify linear and non-linear differential equations.
- Identify homogeneous and non-homogenous differential equations.
- Write input/output equations using derivatives and integrals for electrical and mechanical systems.

Linear Dynamic Systems

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Definition
Linear Differential Equation - a linear combination of derivatives of an unknown function and the unknown function. Derivatives capture how system variables change with time.

Linear systems - represented with linear differential equations

Solving a differential equation means find a function that changes with time that satisfies the equation. The result is a function and not a number. This function describes how a quantity changes with respect to the independent variable, usually time in a control system. This can be done using analytic or numerical methods.

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Linear Dynamic Systems

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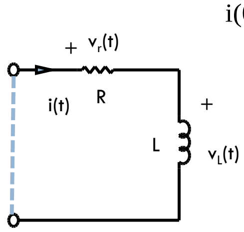
Example 8-1 Series RL circuit with a current established initially. What does current do over time?



$$i(0) = 1\,\text{A}$$

Write KVL equation around the circuit

$$v_R(t) = i(t)\cdot R \quad v_L(t) = L\left(\frac{di(t)}{dt}\right)$$

$$v_L(t) + v_R(t) = 0$$

$$L\left(\frac{di(t)}{dt}\right) + i(t)\cdot R = 0$$

To solve - find i(t) that satisfies the above equation with initial current of 1 A. Can do analytically or numerically.

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Linear Differential Equations

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More Complex Differential Equations

Can have higher order Derivatives 2, 3, 4… etc

$$\frac{d^2}{dt^2}i(t)+2\cdot\frac{d}{dt}i(t)-7\cdot i(t)=0$$

Above is a 2nd order linear ODE (ordinary differential equation)

Implied function of time. i=i(t)

$$\frac{d^2}{dt^2}i+2\cdot\frac{d}{dt}i+i^2=0$$

Squared i makes it non-linear

Above is 2nd order, non-linear ODE

$$\left(\frac{d^2}{dt^2}v\right)\cdot\sin(v)+v=0$$

2nd order, non-linear ODE
Sine of unknown function v(t)

When right-hand side (RHS) is 0, equation called homogeneous. Implies no outside stimulation

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Physical System Examples

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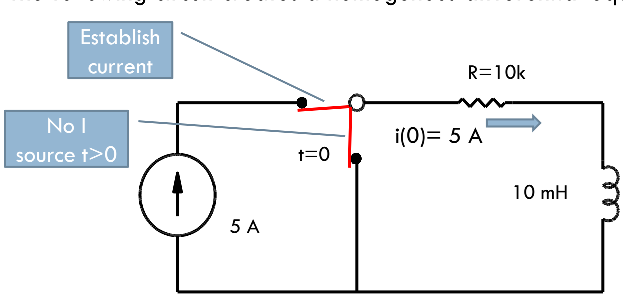
The following circuit creates a homogenous differential equation.



Current can not change instantaneously in inductor. Current varies in time based in the initial value of i(0)= 5A

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Physical System Examples

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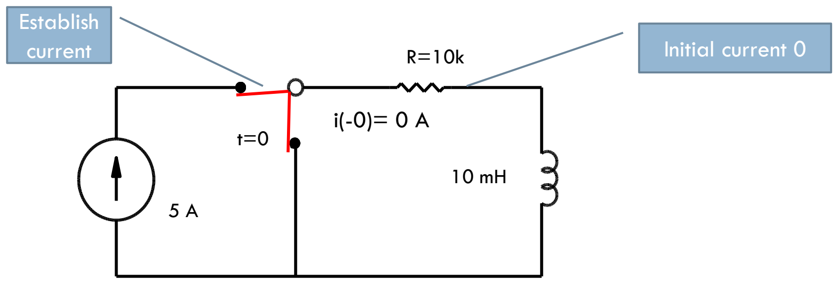
The following circuit creates a non-homogenous differential equation.



Current builds from initial value. Practical example: Energize a relay coil with the current source. Current source drives the system.

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Identify Example Equations

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Identify which of the following equations as linear/non-linear and homogeneous/non-homogeneous.

$$-3\cdot\frac{d}{dt}x(t) + x(t)^2 = 0$$

$$L\cdot\frac{d}{dt}i(t) + R\cdot i(t) = 0$$

$$\left(\frac{d}{dt}v\right)^2 + \sin(v) = 0$$

$$6\cdot\frac{d}{dt}v + 2\cdot v = V_m\cdot\sin(\omega t)$$

$$4\cdot\frac{d^2}{dt^2}i - 2\cdot\frac{d}{dt}i - 7\cdot i - I_o\cdot e^{-\frac{t}{\tau}} = 0$$

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Differential Equations For Control Systems

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Equations have constant coefficients and are linear.
Single input stimulation r(t)
Single output variable x(t)

General form

$$a_n \cdot \frac{d^n}{dt^n} x(t) + \ldots + a_2 \cdot \frac{d^2}{dt^2} x(t) + a_1 \cdot \frac{d}{dt} x(t) + a_1 \cdot x(t) = b_0 \cdot r(t)$$

Where a_n.....a_2, a_1 a_0 and b_0 are constants

What can r(t) be?

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Differential Equations For Control Systems

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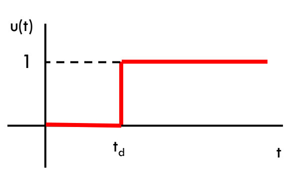
Typical input functions r(t)

Constant
-5
3
10

Ramp
3t
-10t
0.2t-3

Sinusoid
100sin(ωt)
-5cos(ωt+θ)
Where ω and θ are constants

Unit-step (square wave)



$u(t-t_d)$ = 1 after t_d
0 before

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Characteristics of Linear Systems

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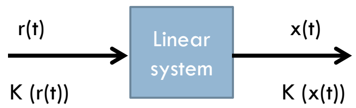
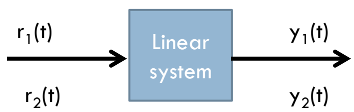
1.) Multiplying by constant is reflected though system.



If input r(t) gives output x(t) then,
K (r(t)) gives K (x(t))
I/O proportional

2.) Superposition from circuits holds



If input $r_1(t)$ gives $y_1(t)$ and input $r_2(t)$ gives $y_2(t)$ then total output $y_1(t)+y_2(t)$

Total output is the sum of the individual input responses. From circuits, transients and sine steady-state

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Dynamic Equations Input/output relationships

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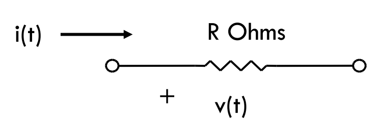
Dynamics represented by integrals and derivatives with respect to time

Electrical element: Resistance



Defining Equations

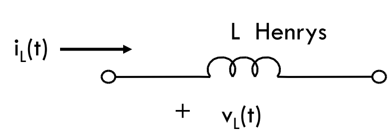
$$v(t) = R \cdot i(t)$$

$$i(t) = \left(\frac{1}{R}\right) \cdot v(t)$$

or

$$i(t) = G \cdot v(t) \quad \text{Where} \quad G = \frac{1}{R}$$

Electrical Element: Inductance



Defining Equations

$$v_L(t) = L \cdot \frac{d}{dt} i_L(t)$$

$$i_L(t) = \frac{1}{L} \cdot \int_0^t v_L(\tau)\, d\tau + i_L(0)$$

Initial current at t=0

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Dynamic Equations Input/output relationships

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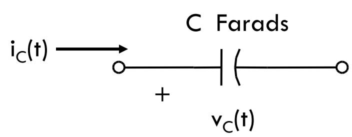
Electrical Element: Capacitance



Defining Equations

$$i_C(t) = C \cdot \frac{d}{dt} v_C(t)$$

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\, d\tau + v_C(0)$$

Initial voltage at t=0

All laws from circuit theory hold for the analysis of circuits with dynamic equations . KCL, KVL, mesh analysis, nodal analysis can all be performed. Substitute the appropriate integral or derivative into the mesh or nodal formulation.

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Dynamic Electrical Equations

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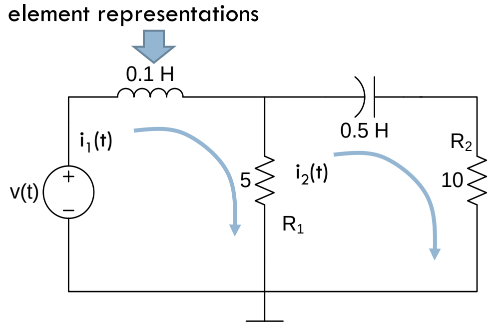
When the current or voltage in a circuit element involves two currents or voltages in the derivative or integral, take the difference of the voltages or currents

Example 8-2: Write mesh equations for the circuit below using the lumped circuit element representations



Loop 1

$$-v(t) + v_L(t) + v_{R1}(t) = 0$$

$$v_L(t) + v_{R1}(t) = v(t)$$

$$L \cdot \frac{d}{dt} i_L(t) + R_1 \cdot (i_1(t) - i_2(t))$$

$$0.1 \cdot \frac{d}{dt} i_1(t) + 5 \cdot (i_1(t) - i_2(t)) = v(t)$$

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Example 8-2 Solution (1)

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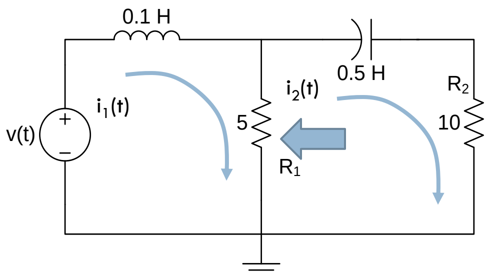

Loop 2

$$v_{R1}(t)+v_C(t)+v_{R2}(t)=0$$

$$\frac{1}{C}\cdot\int_0^t i_2(\tau)\,d\tau+v_C(0)$$

$$5\cdot\left(i_2(t)-i_1(t)\right)\ +\frac{1}{0.5}\cdot\int_0^t i_2(\tau)\,d\tau+v_C(0)\ \ +10\cdot\left(i_2(t)\right)=0$$

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Example 8-2 Solution (2)

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Mesh equations form a system of integral-differential equations in unknown functionsi_1(t) and i_2(t).

Loop 1 $5\cdot\left(i_1(t)-i_2(t)\right)+0.1\cdot\frac{d}{dt}i_1(t)=v(t)$ (1)

1/0.5=2

Loop 2 $5\cdot\left(i_2(t)-i_1(t)\right)+2\cdot\int_0^t i_2(\tau)\,d\tau+v_C(0)+10\cdot i_2(t)=0$ (2)

Solution techniques: convert all equations into derivatives only. Approximate derivatives using mathematical methods and calculate approximate derivative values for some small increment in time. Results are a list of computed points that approximate variable over a time interval. Graph these points to see system response

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Example 8-3 OP AMP Differentiators

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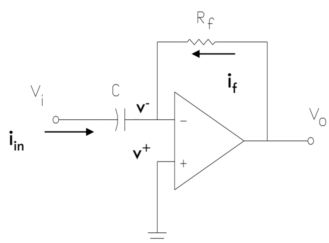
Use rules of circuit analysis and ideal OP AMPs to find the input/output relationship for the circuit below.



Rules of OP AMPS

1.) No current flows into OP AMP

2.) $V^- = V^+$

Use nodal analysis at OP AMP inverting node.

Sum currents at inverting input

$i_{in}(t)+i_f(t)=0$ so $i_{in}(t)=-i_f(t)$

$i_{in}(t)=i_C(t)$

Define $i_c(t)$ in terms of voltage

$$i_C(t) = C \cdot \frac{d}{dt} v_C(t)$$

$$v_{C(t)} = v_{in}(t) - v^-(t)$$

Feedback current

$$i_f(t) = \frac{v_o(t) - v^-(t)}{R_f}$$

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Example 8-3 Solution (2)

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Complete derivation

$$i_{in}(t) = -i_f(t)$$

$$C \cdot \frac{d}{dt}\left[v_{in}(t) - v^-(t)\right] = -\left[\frac{v_o(t) - v^-(t)}{R_f}\right]$$

$V^+(t)=V^-(t)=0$ Positive terminal grounded

$$C \cdot \frac{d}{dt}\left[v_{in}(t)\right] = \left[\frac{-v_o(t)}{R_f}\right]$$

$$-R_f \cdot C \cdot \frac{d}{dt}\left[v_{in}(t)\right] = v_o(t)$$

Input

Output

Circuit takes derivative of input voltage

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Mechanical System Models

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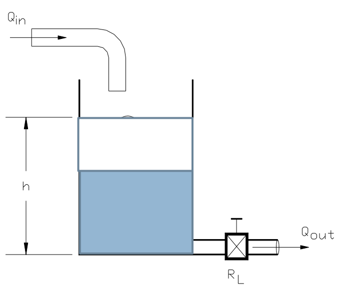
Self-regulating tank system



Need relationship for how level changes with time. Derive differential equation.

$$\Delta V = (Q_{in} - Q_{out}) \cdot \Delta t \quad (1)$$

Write level change in terms of tank volume

$$\Delta h = \frac{\Delta V}{A} \quad (2)$$

$Q_{in} > Q_{out}$ h increases

$Q_{in} < Q_{out}$ h decreases

$$\Delta h = \frac{\Delta V}{A} = \frac{(Q_{in} - Q_{out}) \cdot \Delta t}{A} \quad (1)\ (2)$$

Mechanical System Models

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Self-regulating tank system – Find the average level change over time interval Δt.

$$\frac{\Delta h}{\Delta t} = \frac{\Delta V}{A} = \frac{(Q_{in} - Q_{out}) \cdot \cancel{\Delta t}}{A \ \cancel{\Delta t}}$$

$$\frac{\Delta h}{\Delta t} = \frac{(Q_{in} - Q_{out})}{A}$$

Take limit as time interval goes to zero

$$\lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} = \frac{dh}{dt} = \frac{(Q_{in} - Q_{out})}{A}$$

Write Q_{out} in terms of system parameters

Self-Regulating Tank

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Assume laminar flow for simplicity. Q_{out} determined by pressure at bottom of tank.

$$p = R_L \cdot Q_{out}$$
$$p = \rho \cdot g \cdot h$$

Combine these two equations and solve for Q_{out}

Bring all terms with h or derivative of h to one side

$$\frac{dh}{dt} = \frac{Q_{in} - \frac{\rho \cdot g \cdot h}{R_L}}{A} \rightarrow \frac{dh}{dt} = \frac{Q_{in}}{A} - \frac{\rho \cdot g \cdot h}{R_L \cdot A}$$

$$\left(\frac{R_L \cdot A}{\rho \cdot g}\right) \cdot \frac{dh}{dt} + \left(\frac{\cancel{R_L \cdot} A}{\cancel{\rho \cdot} g}\right) \cdot \left(\frac{\cancel{\rho \cdot} g}{R_L \cancel{\cdot A}}\right) \cdot h = \left(\frac{Q_{in}}{\cancel{A}}\right)\left(\frac{R_L \cdot \cancel{A}}{\rho \cdot g}\right)$$

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Self-Regulating Tank

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Simplified result

$$\left(\frac{R_L \cdot A}{\rho \cdot g}\right) \cdot \frac{dh}{dt} + h = Q_{in}\left(\frac{R_L}{\rho \cdot g}\right)$$

Q_{in} is independent of the liquid height. consider it along with density, area and flow resistance to be constant

Let $\tau = \left(\frac{R_L \cdot A}{\rho \cdot g}\right)$ $\quad G = \left(\frac{R_L}{\rho \cdot g}\right)$

$$\tau \cdot \frac{dh}{dt} + h = G \cdot Q_{in}$$

Non-homogeneous equation that determines how height of liquid in tank varies with time.
Shutting off Q_{in} finds homogeneous response of tank height

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Mechanical Models

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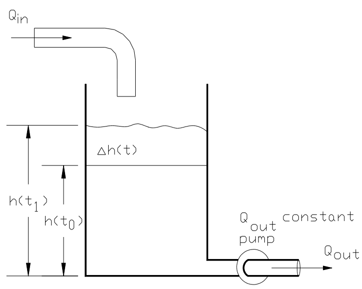
Non-Self-Regulating Tank (Pumped Drainage)



Output flow is fixed by pump flow rate. It is independent of the liquid height in tank.

$$\Delta h = \frac{\Delta V}{A} = \frac{(Q_{in} - Q_{out}) \cdot \Delta t}{A}$$

Define: $\Delta t = t_1 - t_0$

$$\Delta h(t) = h(t_1) - h(t_0)$$

$$\frac{\Delta h}{\Delta t} = \frac{(Q_{in} - Q_{out})}{A} \qquad \lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} = \frac{dh}{dt}$$

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Mechanical Models

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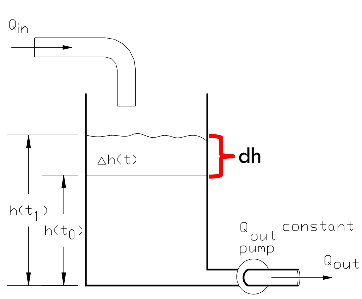
Non-Self-Regulating Tank (Pumped Drainage)



$$\frac{dh}{dt} = \frac{(Q_{in} - Q_{out})}{A}$$

Independent of h

To solve this differential equation, integrate both sides of the equation with respect to t.

$$\int_{t_0}^{t_1} \frac{dh}{dt} dt = \int_{t_0}^{t_1} \frac{1}{A}(Q_{in} - Q_{out}) dt$$

$$h(t_1) - h(t_0) = \int_{t_0}^{t_1} \frac{1}{A}(Q_{in} - Q_{out}) dt$$

$$\Delta h(t) = \boxed{\frac{1}{A}(Q_{in} - Q_{out})} \int_{t_0}^{t_1} dt$$

Note: Right hand side is not a function of t. It is all a constant and can be taken out of the integral.

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Non-Self-Regulating Tank (Pumped Drainage)

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If Q_{in} is not a constant but changes with time, the formula below models the response of the tank system

$$h(t_1) - h(t_0) = \int_{t_0}^{t_1} \frac{1}{A}\left(Q_{in}(t) - Q_{out}\right) dt$$

$$h(t_1) = \int_{t_0}^{t_1} \frac{1}{A}\left(Q_{in}(t) - Q_{out}\right) dt + h(t_0)$$

Definite integral from calculus

Initial tank height

Final tank height

For pumped tank with constant input and output flows, tank drains linearly with time based on the difference between the flow rates. Final tank height depends on the pump flow rate and the time the pump operates.

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Thermal Systems

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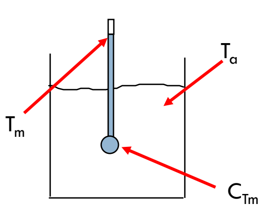
Heating characteristic of a liquid filled thermometer



T_a = fluid temperature
T_m = measured temperature
C_{Tm} = thermal capacitance of thermometer

How does measured temperature change with time? Heat transferred to thermometer depends on ΔT, R_T and time interval

$$\Delta Q = \frac{(T_a - T_m) \cdot \Delta t}{R_T}$$ Incremental conduction heat flow

Definition of thermal capacitance

$$\frac{\Delta Q}{\Delta T_m} = C_{Tm} \rightarrow \frac{\Delta Q}{C_{Tm}} = \Delta T_m$$

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Thermal Systems

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Derive the thermal equation

$$\frac{\Delta Q}{C_{Tm}} = \frac{(T_a - T_m)\cdot \Delta t}{R_T \cdot C_{Tm}}$$ Divide both sides by C_{Tm}

$$\Delta T_m = \frac{(T_a - T_m)\cdot \Delta t}{R_T \cdot C_{Tm}}$$ Use definition of C_{Tm} from last slide

Determine the average change in temperature for a Δt

$$\frac{\Delta T_m}{\Delta t} = \frac{(T_a - T_m)}{R_T \cdot C_{Tm}}$$

Take limit Δt approaches 0 $$\lim_{\Delta t \to 0} \frac{\Delta T_m}{\Delta t} = \frac{dT_m}{dt} = \frac{(T_a - T_m)}{R_T \cdot C_{Tm}}$$

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Thermal Systems

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Get all T_m and its derivatives on one side of the equation

$$\frac{dT_m}{dt} = \frac{(T_a - T_m)}{R_T \cdot C_{Tm}}$$

$$\frac{dT_m}{dt} = \frac{T_a}{R_T \cdot C_{Tm}} - \frac{T_m}{R_T \cdot C_{Tm}}$$

$$\frac{dT_m}{dt} + \frac{T_m}{R_T \cdot C_{Tm}} = \frac{T_a}{R_T \cdot C_{Tm}}$$ Move T_m to same side as derivative of T_m

$$R_T \cdot C_{Tm} \cdot \frac{dT_m}{dt} + \frac{T_m}{\cancel{R_T \cdot C_{Tm}}} \cdot (\cancel{R_T \cdot C_{Tm}}) = \frac{T_a}{\cancel{R_T \cdot C_{Tm}}} \cdot (\cancel{R_T \cdot C_{Tm}})$$

$$R_T \cdot C_{Tm} \cdot \frac{dT_m}{dt} + T_m = T_a$$

This is similar in form to the self-regulating tank equation. This is a non-homogeneous differential equation that describes how the measured temperature changes with time

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Mechanical Systems

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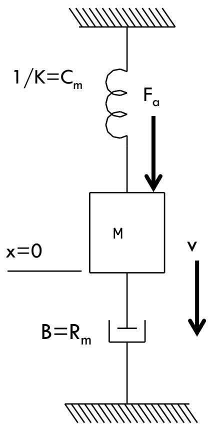
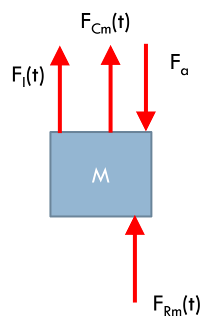
Pneumatic Control Value Position



Determine how the position of a air actuated control value changes with time after pressure is applied.

Free body diagram- All forces must sum to zero



$F_a = P_a(A)$ = input air force

All forces must balance at each instance in time so:

$F_I(t)$ = inertial force

$F_{Cm}(t)$ = spring force

$F_{Rm}(t)$ = viscous friction force

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Mechanical Systems- Control Valves

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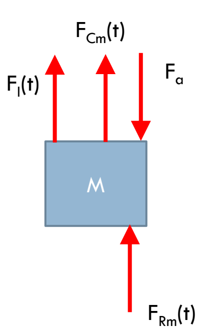
All forces must sum to zero- Assume down is positive direction



$$F_a - F_{Rm}(t) - F_I(t) - F_{Cm}(t) = 0$$

$$F_a = F_{Rm}(t) + F_I(t) + F_{Cm}(t)$$

Need equation that relates position, x, to time

Friction force

$$F_{Rm}(t) = R_m \cdot v(t) \quad \text{Remember} \quad v(t) = \frac{dx(t)}{dt}$$

$$F_{Rm}(t) = R_m \cdot \frac{dx(t)}{dt}$$

Viscous friction is proportional to velocity. Velocity = rate of change of position

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Mechanical Systems- Control Valves

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Spring force

$$F_{Cm}(t) = \left(\frac{1}{C_m}\right)\cdot x(t) = K \cdot x(t)$$

Force from spring is proportional to its length, x. C_m is spring capacitance, K = spring constant. So $1/K = C_m$.

Inertial force

$$F_I(t) = M \cdot a(t)$$

$$a(t) = \frac{dv(t)}{dt} \qquad v(t) = \frac{dx(t)}{dt}$$

$$a(t) = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}$$

$$F_I(t) = M \cdot \frac{d^2x(t)}{dt^2}$$

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Control Valve Model

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Combine individual terms

$$F_a = F_{Rm}(t) + F_I(t) + F_{Cm}(t)$$

$$F_a = R_m \cdot \frac{dx(t)}{dt} + M \cdot \frac{d^2x(t)}{dt^2} + \left(\frac{1}{C_m}\right)\cdot x(t)$$

F_a is constant. Equation describes how position changes with time. Second order equation, non-homogeneous.

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Summary of Mechanical Models

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Self-Regulating Tank

$$\tau \cdot \frac{dh}{dt} + h = G \cdot Q_{in}$$

$$\tau = \left(\frac{R_L \cdot A}{\rho \cdot g}\right) \qquad G = \left(\frac{R_L}{\rho \cdot g}\right)$$

Non-Self-Regulating Tank (Pumped Drainage)

$$h(t_1) = \int_{t_0}^{t_1} \frac{1}{A}\left(Q_{in}(t) - Q_{out}\right)dt + h(t_0)$$

Heat Transfer

$$R_T \cdot C_{Tm} \cdot \frac{dT_m}{dt} + T_m = T_a$$

Control Valve Position

$$F_a = M \cdot \frac{d^2x(t)}{dt^2} + R_m \cdot \frac{dx(t)}{dt} + \left(\frac{1}{C_m}\right) \cdot x(t)$$

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End Lesson 8: Modeling Physical Systems With Linear Differential Equations

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